# Analysis I - Serie 04
# FSU Jena - WS 06/07
# - Lösungen -

Stilianos Louca

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## Aufgabe 1

a)

$$\left(1+\frac{1}{x}\right)^x : \textit{streng monoton wachsend.}$$

$$\Rightarrow \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor} \leq \left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{\lceil x\rceil}\right)^{\lceil x\rceil}$$

$$\Rightarrow e = \lim_{\lfloor x\rfloor\to\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor} \leq \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \leq \lim_{\lceil x\rceil\to\infty}\left(1+\frac{1}{\lceil x\rceil}\right)^{\lceil x\rceil} = e$$

$$\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \quad \square$$

b)

$$\lim_{x\to\infty}\frac{x}{x-1} = 1 = \lim_{x\to\infty}\frac{x+1}{x} \Rightarrow \lim_{x\to\infty}\left(\frac{x}{x-1}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x$$

$$\Rightarrow \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{-x} = \lim_{x\to\infty}\left(\frac{1}{1-\frac{1}{x}}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\Rightarrow \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y = \lim_{(-y)\to\infty}\left(1-\frac{1}{-y}\right)^{-(-y)} = e$$

c)

$$\lim_{\varepsilon\to 0}(1+\varepsilon))^{\frac{1}{\varepsilon}} = \lim_{|\frac{1}{\varepsilon}|\to\infty}\left(1+\frac{1}{\frac{1}{\varepsilon}}\right)^{\frac{1}{\varepsilon}} = e$$

*Extra :*

$$\forall a\in\mathbb{R}^\times : \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{|\frac{x}{a}|\to\infty}\left(\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right)^a = \left(\lim_{|t|\to\infty}\left(1+\frac{1}{t}\right)^t\right)^a = e^a$$

## Aufgabe 2

Zeigen: $\exists \lim a_n$.

$$a_{n+1} = \sqrt{2+a_n},\ a_1 = \sqrt{2} < 2,\ a_2 > a_1$$

$$i)\ Ist\ a_n < 2\ \Rightarrow a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2\ \Rightarrow\ a_n < 2\ \forall\ n \in \mathbb{N}$$

$$ii)\ Ist\ a_n > a_{n-1}\ \Rightarrow\ a_{n+1} = \sqrt{2+a_n} > \sqrt{2+a_{n-1}} = a_n\ \Rightarrow\ a_{n+1} > a_n\ \forall\ n \in \mathbb{N}$$

$$\Rightarrow\ \exists\ \lim(a_n) =: a > 0$$

Berechnung von $\lim a_n$:

$$a = \lim a_n = \lim \sqrt{2+a_{n-1}} = \sqrt{2+\lim a_{n-1}} = \sqrt{2+a}\ \Rightarrow\ (a-2)(a+1) = 0\ \Rightarrow\ a = 2$$

## Aufgabe 3

Zeigen: $\exists\ \lim x_n$.

$$i)\ \left(x_n > \sqrt{a}\right)\ \Leftrightarrow\ a < x_n^2\ \Leftrightarrow\ \frac{a}{x_n} < x_n\ \Leftrightarrow\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) < x_n$$

$$ii)\ \left(x_n = \sqrt{a}\right)\ \Leftrightarrow\ x_{n+1} = \sqrt{a}$$

$$iii)\ (i)\ \Rightarrow\ \left(\left(x_n < \sqrt{a}\right)\ \Rightarrow \left(x_{n+1} > \sqrt{a} > x_n\right)\right)$$

$$\Rightarrow\ \forall\ 1 < n \in \mathbb{N} : (a_n)\ monoton\ fallend.$$

$$d := x_n - \sqrt{a}$$

$$x_{n+1} - \sqrt{a} = \frac{1}{2}\left(\sqrt{a} + d + \frac{a}{\sqrt{a}+d}\right) - \sqrt{a} = \frac{d^2}{2\left(\sqrt{a}+d\right)} > 0\ \Rightarrow\ \forall\ 1 < n \in \mathbb{N} : x_n > \sqrt{a}$$

$$\Rightarrow\ \forall\ 1 < n \in \mathbb{N} : (x_n)\ nach\ unten\ beschränkt\ \wedge\ monoton\ fallend\ \Rightarrow \exists\ \lim x_n =: b \geq \sqrt{a}$$

Berechnung von $\lim x_n$:

$$b = \lim x_n = \lim \frac{1}{2}\left(1 + \frac{a}{x_n}\right) = \frac{1}{2}\left(\lim x_n + \frac{a}{\lim x_n}\right) = \frac{1}{2}\left(b + \frac{a}{b}\right)\ \Rightarrow \lim x_n = b = \sqrt{a}$$

## Aufgabe 4

$$a_n := \frac{5n+6}{n+1}$$

$$\textit{Sei } \varepsilon > 0 \textit{ beliebig. Dann}:$$

$$\exists\ n_0 \in \mathbb{N} : n_0 > \frac{1}{\varepsilon} - 1 \ \Rightarrow\ \forall\ n > n_0 : 0 < \frac{1}{n+1} < \varepsilon$$

$$\Rightarrow\ \forall\ n > n_0\ :\ |a_n - 5| = \left|\frac{1}{n+1}\right| < \varepsilon\ \Rightarrow \lim a_n = 5 \quad \square$$

## Aufgabe 5

a)

$$b_n := a\ \Rightarrow\ \lim \sqrt{a} = \lim \sqrt{b_n} = \lim \frac{b_{n+1}}{b_n} = \frac{a}{a} = 1$$

b)

$$\lim \sqrt[n]{n} = \lim \frac{n+1}{n} = 1$$

c)

$$\lim \frac{1}{n} \cdot \sqrt[n]{n!} = \lim \sqrt[n]{\frac{n!}{n^n}} = \lim \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim \left(\frac{n}{n+1}\right)^n = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

d)

$$Aufgabe\ 01 \rightarrow \lim_{n\to\infty} \left(1 + \frac{-7}{n}\right)^n = \frac{1}{e^7}$$